Fischer Decomposition for Difference Dirac Operators

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February 2, 2008

Abstract

We establish the basis of a discrete function theory starting with a Fischer decomposition for difference Dirac operators. Discrete versions of homogeneous polynomials, Euler and Gamma operators are obtained. As a consequence we obtain a Fischer decomposition for the discrete Laplacian.

MSC 2000: 30G35, 30G25, 39A12, 39A70
keywords: Difference Dirac operator, Difference Euler operator, Fischer decomposition

1 Introduction

Clifford analysis is a powerful tool to solve some kinds of problems related with vector field analysis.
A comprehensive description of Clifford function theory was introduced by F.

*supported by *Fundação para Ciência e Tecnologia* under PhD-grant no. SFRH/BD/17657/2004

†Research (partially) supported by *Unidade de Investigação Matemática e Aplicações* of the University of Aveiro.

Brackx, R. Delanghe and F. Sommen in [1] and later by R. Delanghe, F. Sommen and V. Souček in [2].

In [5, 6], K. Gürlebeck and W. Sprößig proposed some strategies to solve boundary value problems based on the study of existence, uniqueness, representation, and regularity of solutions with the help of an operator calculus. In the same books, the authors introduce also the basic ideas to develop a discrete counterpart of the continuous treatment of boundary value problems with the introduction of a discrete operator calculus in order to find a well-adapted numerical approach. An explicit discrete version of the Borel-Pompeiu formula was presented for dimension $n = 3$.

This was further developed in [7, 8], where K. Gürlebeck and A. Hommel developed finite difference potential methods in lattice domains based on the concept of discrete fundamental solutions for the difference Dirac operator which generalizes the work developed by Ryabenkij in [10]. A numerical application of this theory was presented recently by N. Faustino, K. Gürlebeck, A. Hommel, and U. Kähler in [3] for the incompressible stationary Navier-Stokes equations. In this paper, the authors proposed a scheme which solves efficiently problems in unbounded domains and show the convergence of the numerical scheme for functions with Hölder regularity which is a better gain compared with the convergence results for classical difference schemes.

Moreover, while all these papers claim to be based on discrete function theoretical approaches, from the concepts of the theory of monogenic functions only the Borel-Pompeiu formula and with it Cauchy's integral formula were obtained. There is no "real" development of a discrete monogenic function theory up to now.

This paper is supposed to be a step in this direction. To this end discrete versions of a Fischer decomposition, Euler and Gamma operators are obtained. For the sake of simplicity we consider in the first part only Dirac operators which contain only forward or backward finite differences. Of course, these Dirac operators do not factorize the classic discrete Laplacian. Therefore, we will consider in the last chapter a different definition of a difference Dirac operator in the quaternionic case (c.f. [7]) which do factorizes the discrete Laplacian.

Let us emphasize in the end a major obstacle in the discrete case. While in the continuous case the are only one partial derivative for each coordinate x_j we have two finite differences in the discrete case. Therefore, we will have not only one Euler or Gamma operator as in the continuous case, but several. Each one will turn out to be connected to one particular Dirac operator.

2 Preliminaries

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis of $\mathbb{R}^n$. The Clifford algebra $C\ell_{0,n}$ is the free algebra over $\mathbb{R}^n$ generated modulo the relation

$$x^2 = -|x|^2\mathbf{e}_0,$$

where $\mathbf{e}_0$ is the identity of $C\ell_{0,n}$. For the algebra $C\ell_{0,n}$ we have the anti-commutation relationship

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\mathbf{e}_0,$$

where δ_{ij} is the Kronecker symbol. In the following we will identify the Euclidean space $\mathbb{R}^n$ with $\bigwedge^1 C\ell_{0,n}$, the space of all vectors of $C\ell_{0,n}$. This means that each element x of $\mathbb{R}^n$ may be represented by

$$x = \sum_{i=1}^{n} x_i\mathbf{e}_i.$$

From an analysis viewpoint one extremely crucial property of the algebra $C\ell_{0,n}$ is that each non-zero vector $x \in \mathbb{R}^n$ has a multiplicative inverse given by $\frac{-x}{|x|^2}$. Up to a sign this inverse corresponds to the Kelvin inverse of a vector in Euclidean space. Moreover, given a general Clifford number $a = \sum_A \mathbf{e}_A a_A$, $A \subset \{1, \ldots, n\}$ we denote by $\mathrm{Sc}\, a = a_\emptyset$ the scalar part and by $\mathrm{Vec}\, a = \mathbf{e}_1 a_1 + \ldots + \mathbf{e}_n a_n$ the vector part.

For all what follows let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$. Then any function $f : \Omega \mapsto C\ell_{0,n}$ has a representation $f = \sum_A \mathbf{e}_A f_A$ with $\mathbb{R}$-valued components f_A. We now introduce the Dirac operator $D = \sum_{i=1}^n \mathbf{e}_i \frac{\partial}{\partial x_i}$. This operator is a hypercomplex analogue to the complex Cauchy-Riemann operator. In particular we have that $D^2 = -\Delta$, where Δ is the Laplacian over $\mathbb{R}^n$. A function $f : \Omega \mapsto C\ell_{0,n}$ is said to be *left-monogenic* if it satisfies the equation $(Df)(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right-monogenic functions. Basic properties of the Dirac operator and left-monogenic functions can be found in [1], [2], [6], and [5].

Now, we need some more facts for our discrete setting. To discretize pointwise the partial derivatives $\frac{\partial}{\partial x_i}$ in the equidistant lattice with mesh width $h > 0$, $\mathbb{R}^n_h = \{mh = (m_1 h, \ldots, m_n h) : m \in \mathbb{Z}^n\}$, we introduce forward/backward differences $\partial_h^{\pm i}$:

$$\partial_h^{\pm i} u(mh) = \mp \frac{u(mh) - u(mh \pm h\mathbf{e}_i)}{h} \tag{1}$$

These forward/backward differences $\partial_h^{\pm i}$ satisfy the following product rules

$$\begin{aligned}
(\partial_h^{\pm i} fg)(mh) &= f(mh)(\partial_h^{\pm i} g)(mh) + (\partial_h^{\pm i} f)(mh) g(mh \pm h\mathbf{e}_i), & (2)\\
(\partial_h^{\pm i} fg)(mh) &= f(mh \pm h\mathbf{e}_i)(\partial_h^{\pm i} g)(mh) + (\partial_h^{\pm i} f)(mh) g(mh). & (3)
\end{aligned}$$

The forward/backward discretizations of the Dirac operator are given by

$$D_h^{\pm} = \sum_{i=1}^{n} \mathbf{e}_i \partial_h^{\pm i}. \tag{4}$$

In this paper we will also use the following multi-index abbreviations:

$$(mh)^{(\alpha)} := (m_1 h)^{\alpha_1} (m_2 h)^{\alpha_2} \ldots (m_n h)^{\alpha_n};$$

$$\begin{aligned}\alpha! &:= \alpha_1!\alpha_2!\ldots\alpha_n!;\\ |\alpha| &:= \alpha_1+\alpha_2+\ldots\alpha_n\\ \partial_h^{\pm\mathbf{e}_i} &:= \partial_h^{\pm i};\\ \partial_h^{\pm\alpha_i\mathbf{e}_i} &:= \left(\partial_h^{\pm\mathbf{e}_i}\right)^{\alpha_i},\end{aligned}$$

for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^n \mathbf{e}_i\alpha_i$.

3 Fischer decomposition

The basic idea of a Fischer decomposition is to decompose any homogeneous polynomial into monogenic homogeneous polynomials of lower degrees. In the classic case such a decomposition is based on the fact that the powers x^s are homogeneous and that $\frac{\partial x_i^s}{\partial x_i} = s x_i^{s-1}$. A first idea would be to consider instead of x^s simply the powers $(mh)^s$, but while these powers are still homogeneous the last condition is not true in the discrete case, unfortunately. Therefore, we will start by introducing discrete homogeneous powers which will play the equivalent role of x^s in the discrete case.

3.1 Multi-index factorial powers

Starting from the one-dimensional factorial powers

$$(m_ih)_{\mp}^{(0)} := 1, (m_ih)_{\mp}^{(s)} := \prod_{k=0}^{s-1}(m_ih \mp kh), s \in \mathbb{N} \tag{5}$$

we introduce the multi-index factorial powers of degree $|\alpha|$ by

$$(mh)_{\mp}^{(\alpha)} = \prod_{i=1}^{n}(m_ih)_{\mp}^{(\alpha_i)}.$$

The one-dimensional factorial powers $(m_ih)_{\mp}^{(s)}$ have the following properties

P1. $(m_ih)_{\mp}^{(s+1)} = (m_ih \mp sh)(m_ih)_{\mp}^{(s)}$;

P2. $\partial_h^{\pm j}(m_ih)_{\mp}^{(s)} = s(m_ih)_{\mp}^{(s-1)}\delta_{i,j}$;

P3. $\partial_h^{\mp j}(m_ih)_{\mp}^{(s)} = s(m_ih \mp h)_{\mp}^{(s-1)}\delta_{i,j}$;

P4. $(m_ih)_{\mp}^{(s)} \to x_i^s = (m_ih)^s$ for $h \to 0$,

where $\delta_{i,j}$ denotes the standard Kronecker symbol.

As a direct consequence of these properties, we obtain the following lemmas:

Lemma 3.1 *The multi-index factorial powers of degree* $|\alpha|$, $(mh)_{\mp}^{(\alpha)}$, *satisfy*

$$\sum_{i=1}^{n}(m_i h)\partial_h^{\pm i}(mh \mp h\mathbf{e}_i)_{\mp}^{(\alpha)} = |\alpha|(mh)_{\mp}^{(\alpha)}.$$

Lemma 3.2 *The multi-index factorial powers of degree* $|\alpha|$, $(mh)_{\mp}^{(\alpha)}$, *satisfy*

$$\partial_h^{\pm\beta}(mh)_{\mp}^{(\alpha)} = \alpha!\delta_{\alpha,\beta} \qquad \text{for } |\beta| = |\alpha|.$$

Lemma 3.3 *The multi-index factorial powers* $(mh)_{\mp}^{(\alpha)}$ *of degree* $|\alpha|$ *approximate the classical multi-index powers* $x^{(\alpha)}$ *of degree* $|\alpha|$ *at each point* $x = mh$.

Let us remark that we have the following relationships between the multi-index factorial powers and the usual powers:

Theorem 3.1 *The powers* $(m_i h)^{\alpha_i}$ *and* $(m_i h)_{\mp}^{\alpha_i}$ *are related by*

$$(m_i h)_{\mp}^{\alpha_i} = \sum_{k_i=0}^{\alpha_i} S_{k_i}^{\alpha_i}(m_i h)^{k_i}$$

$$(m_i h)^{\alpha_i} = \sum_{k_i=0}^{\alpha_i} T_{k_i}^{\alpha_i}(m_i h)_{\mp}^{k_i},$$

where $S_{k_i}^{\alpha_i}$ *are the Stirling numbers of the first kind and* $T_{k_i}^{\alpha_i}$ *are the Stirling numbers of the second kind.*

The sketch of the proof of this theorem can be found, e.g., in [11].

Theorem 3.2 *The multi-index powers* $(mh)^{(\alpha)}$ *and* $(mh)_{\mp}^{(\alpha)}$ *are related by*

$$(mh)_{\mp}^{(\alpha)} = \sum_{|\beta|=0}^{|\alpha|} K_{\beta}^{\alpha}(mh)^{(\beta)}, \tag{6}$$

$$(mh)^{(\alpha)} = \sum_{|\beta|=0}^{|\alpha|} L_{\beta}^{\alpha}(mh)_{\mp}^{(\beta)}. \tag{7}$$

Moreover,

$$K_{\beta}^{\alpha} = \sum_{l_{n-1}=0}^{|\beta|}\sum_{l_{n-2}=0}^{l_{n-1}}\cdots\sum_{l_1=0}^{l_2} S_{l_1}^{\alpha_1}S_{l_2-l_1}^{\alpha_2}\cdots S_{|\beta|-l_{n-1}}^{\alpha_n},$$

$$L_{\beta}^{\alpha} = \sum_{l_{n-1}=0}^{|\beta|}\sum_{l_{n-2}=0}^{l_{n-1}}\cdots\sum_{l_1=0}^{l_2} T_{l_1}^{\alpha_1}T_{l_2-l_1}^{\alpha_2}\cdots T_{|\beta|-l_{n-1}}^{\alpha_n}.$$

We will just prove identity (6). The proof of identity (7) is analogous to the proof of identity (6).

Proof: Using Theorem 3.1 and multiplying the polynomials $(m_1h)^{\alpha_1}$ and $(m_2h)^{\alpha_2}$, we obtain

$$(m_1h)^{\alpha_1}(m_2h)^{\alpha_2} = \sum_{\beta_1+\beta_2=0}^{\alpha_1+\alpha_2} K_{\alpha_1,\alpha_2}^{\beta_1,\beta_2}(m_1h)_{\mp}^{(\beta_1)}(m_2h)_{\mp}^{(\beta_2)}$$

with $K_{\alpha_1,\alpha_2}^{\beta_1,\beta_2} = \left(\sum_{l_1=0}^{\beta_1+\beta_2} S_{l_1}^{\alpha_1} S_{\beta_1+\beta_2-l_1}^{\alpha_2}\right)$.

Using again Theorem 3.1 and multiplying the polynomials $(m_1h)^{\alpha_1}(m_2h)^{\alpha_2}$ and $(m_3h)^{\alpha_3}$, we obtain

$$(m_1h)^{\alpha_1}(m_2h)^{\alpha_2}(m_3h)^{\alpha_3} = \sum_{\beta_1+\beta_2+\beta_3=0}^{\alpha_1+\alpha_2+\alpha_3} K_{\alpha_1,\alpha_2,\alpha_3}^{\beta_1,\beta_2,\beta_3}(m_1h)_{\mp}^{(\beta_1)}(m_2h)_{\mp}^{(\beta_2)}(m_3h)_{\mp}^{(\beta_3)}$$

with $K_{\alpha_1,\alpha_2,\alpha_3}^{\beta_1,\beta_2,\beta_3} = \sum_{l_2=0}^{\beta_1+\beta_2+\beta_3} \sum_{l_1=0}^{l_2} S_{l_1}^{\alpha_1} S_{l_2-l_1}^{\alpha_2} S_{\beta_1+\beta_2+\beta_3-l_2}^{\alpha_3}$.

Applying this procedure recursively, we obtain

$$(mh)^{(\alpha)} = \sum_{|\beta|=0}^{|\alpha|} K_{\beta}^{\alpha}(mh)_{\mp}^{(\alpha)} \tag{8}$$

with $K_{\beta}^{\alpha} = \sum_{l_{n-1}=0}^{|\beta|} \sum_{l_{n-2}=0}^{l_{n-1}} \ldots \sum_{l_1=0}^{l_2} S_{l_1}^{\alpha_1} S_{l_2-l_1}^{\alpha_2} \ldots S_{|\beta|-l_{n-1}}^{\alpha_n}$. ■

For all what follows, let $\Pi_k^{\pm}$ denote the space of all Clifford-valued polynomials of degree k, $P_k^{\pm}$, generated by the powers $(mh)_{\mp}^{(\alpha)}$ of degree $|\alpha| = k$, and $\Pi^{\pm}$ be the countable union of all Clifford-valued polynomials of degree $k \geq 0$. Furthermore, let $\mathcal{M}_k^{\pm} = \Pi_k^{\pm} \cap \ker D_h^{\pm}$ be the space of discrete monogenic polynomials of degree k. Based on Lemma 3.1, 3.2 and 3.3, we will show that it is possible to obtain discrete versions for the Fischer decomposition as well as define discrete versions of the Euler and Gamma operators.

3.2 The main theorem

For two Clifford-valued polynomials of degree k, $P_k^{\pm}$ and $Q_k^{\pm} \in \Pi_k^{\pm}$ given by

$$\begin{aligned} P_k^{\pm}(mh) &= \sum_{|\alpha|=k} (mh)_{\mp}^{(\alpha)} a_{\alpha}^{\pm} \\ Q_k^{\pm}(mh) &= \sum_{|\alpha|=k} (mh)_{\mp}^{(\alpha)} b_{\alpha}^{\pm} \end{aligned}$$

we define the Fischer inner product by

$$[P_k^{\pm}, Q_k^{\pm}]_h := \sum_{|\alpha|=k} \alpha! \mathrm{Sc}\,(\overline{a_{\alpha}^{\pm}} b_{\alpha}^{\pm}). \tag{9}$$

Denoting by $P_k^\pm(D_h^\pm)$ the difference operator obtained from the polynomial $P_k^\pm$ in powers of mh by replacing $m_i h$ by the difference operator $\partial_h^{\pm i}$ (just like in the continuous case, c.f. [2]), we have by Lemma 3.2 the identity

$$[P_k^\pm, Q_k^\pm]_h := \mathrm{Sc}\,(\overline{P_k^\pm(D_h^\pm)}Q_k^\pm)(0) \qquad P_k^\pm, Q_k^\pm \in \Pi_k^\pm. \tag{10}$$

With other words, we can express the Fischer inner product by applying the difference operator $P_k^\pm(D_h^\pm)$ to the polynomial $P_k^\pm$ and evaluate the scalar part at the point $mh = 0$.

Moreover, due to $\overline{D_h^\pm} = -D_h^\pm$ the Fischer inner product has the important property:

$$[(mh)P_k^\pm, Q_k^\pm]_h = -[P_k^\pm, D_h^\pm Q_k^\pm]_h. \tag{11}$$

This property allows us to prove the following theorem:

Theorem 3.3 *We have*

$$\Pi_k^\pm = \mathcal{M}_k^\pm + (mh)\Pi_{k-1}^\pm.$$

Moreover, the subspaces $\mathcal{M}_k^\pm$ and $(mh)\Pi_{k-1}^\pm$ are orthogonal with respect to the Fischer inner product.

Before we prove Theorem 3.3, we will prove the following inclusion property:

Lemma 3.4 *For the set $D_h^\pm\Pi_k^\pm := \{D_h^\pm P_k^\pm : P_k^\pm \in \Pi_k^\pm\}$, we have the inclusion:*

$$D_h^\pm\Pi_k^\pm := \{D_h^\pm P_k^\pm : P_k^\pm \in \Pi_k^\pm\} \subset \Pi_{k-1}^\pm.$$

Proof:

Let $P_k^\pm(mh) = \sum_{|\alpha|=k}(mh)_\mp^{(\alpha)}a_\alpha^\pm \in \Pi_k^\pm$. Applying $D_h^\pm$, we obtain from $\partial_h^{\pm i}(mh)_\mp^{(\alpha)} = \alpha_i(mh)_\mp^{(\alpha-\mathbf{e}_i)}$, the identity

$$(D_h^\pm P_k^\pm)(mh) = \sum_{i=1}^{n}\sum_{|\alpha|=k}(mh)_\mp^{(\alpha-\mathbf{e}_i)}\alpha_i\mathbf{e}_i a_\alpha^\pm \tag{12}$$

Because $\alpha_i\mathbf{e}_i a_\alpha^\pm$ is a Clifford constant we have a linear combination of polynomials of degree $|\alpha - \mathbf{e}_i| = k-1$ on the right hand side of (12). Hence, $D_h^\pm P_k^\pm \in \Pi_{k-1}^\pm$. ■

Proof: (**Theorem 3.3**) Because of $\Pi_k^\pm = (mh)\Pi_{k-1}^\pm + \big((mh)\Pi_{k-1}^\pm\big)^\perp$ it is enough to prove that $\big((mh)\Pi_{k-1}^\pm\big)^\perp = \mathcal{M}_{k-1}^\pm$. For this we choose $P_{k-1}^\pm \in \Pi_{k-1}^\pm$ arbitrarily and assume that for some $P_k^\pm \in \Pi_k^\pm$ we have

$$[(mh)P_{k-1}^\pm, P_k^\pm]_h = 0.$$

Due to (9) we have $[P_{k-1}^\pm, D_h^\pm P_k^\pm]_h = 0$ for all $P_{k-1}^\pm$. As $D_h^\pm P_k^\pm \in \Pi_{k-1}^\pm$ by Lemma 3.4, we obtain $D_h^\pm P_k^\pm = 0$ or $P_k^\pm \in \mathcal{M}_k^\pm$. This means that $((mh)\Pi_{k-1}^\pm)^\perp \subset \mathcal{M}_k^\pm$. Now, let $P_k^\pm \in \mathcal{M}_k^\pm$. Then we have for each $P_{k-1}^\pm \in \Pi_{k-1}^\pm$

$$\begin{aligned}[(mh)P_{k-1}^\pm, P_k^\pm]_h &= -[P_{k-1}^\pm, D_h^\pm P_k^\pm]_h \\ &= 0\end{aligned}$$

and, therefore, $((mh)\Pi_{k-1}^{\pm})^{\top} = \mathcal{M}_{k-1}^{\pm}$. ■

From this theorem we obtain the Fischer decomposition with respect to our difference Dirac operators $D_h^{\pm}$.

Theorem 3.4 Fischer decomposition *Let $P_k^{\pm} \in \Pi_k^{\pm}$ then*

$$P_k^{\pm}(mh) = \sum_{s=0}^{k-1} (mh)^s M_{k-s}^{\pm}(mh). \tag{13}$$

where each $M_j^{\pm}$ denotes the homogeneous discrete monogenic polynomials of degree j with respect to the Dirac operators $D_h^{\pm}$.

3.3 Difference Euler and Gamma operators

Based on Lemma 3.1 we will introduce discrete versions of the Euler and Gamma operators presented in [2].

First of all, we introduce the second order difference operator $A_h^{\pm}$ by

$$A_h^{\pm} = \mp h \sum_{i=1}^{n} (m_i h) \partial_h^{\pm i} \partial_h^{\mp i}. \tag{14}$$

Definition 3.1 *For a lattice function $f_h : \mathbb{R}_h^n \to C\ell_{0,n}$, we introduce the difference Euler operator $E_h^{\pm}$ by*

$$(E_h^{\pm} f_h)(mh) = \sum_{i=1}^{n} (m_i h)(\partial_h^{\pm i} f_h)(mh \mp h\mathbf{e}_i)$$

and the difference Gamma operator $\Gamma_h^{\pm}$ by

$$(\Gamma_h^{\pm} f_h)(mh) = -\sum_{j<k} \mathbf{e}_j \mathbf{e}_k (L_{jk}^{\pm} f_h)(mh) - (A_h^{\pm} f_h)(mh),$$

where $L_{jk}^{\pm} := (m_j h)\partial_h^{\pm k} - (m_k h)\partial_h^{\pm j}$.

It looks surprising that we have in the definition of the Gamma operator a term which contains second order differences, but we would like to remark that for $h \to 0$ this term vanishes and we will get the usual continuous Gamma operator. As a matter of fact this term arises due to the fact that in the discrete case translations are involved in the definition of finite differences/ finite difference operators.

Using the definition of the difference Euler operator and Lemma 3.1, we obtain for polynomials homogeneous of degree k, $P_k^{\pm} \in \Pi_k^{\pm}$, $E_h^{\pm} P_k^{\pm} = k P_k^{\pm}$, and, moreover, we can show that a function f_h homogeneous of degree k satisfy $E_h^{\pm} f_h = k f_h$. This fact provides a good motivation for calling $E_h^{\pm}$ Euler operator, i.e. an operator who measures the degree of homogeneity of a homogeneous function.

It follows from the definition of the Euler and Gamma operator that

$$
\begin{aligned}
(mh)D_h^{\pm} f_h &= -\sum_{i=1}^{n}(m_i h)\partial_h^{\pm i} f_h + \sum_{j<k} \mathbf{e}_j \mathbf{e}_k L_{jk}^{\pm} f_h && (15)\\
&= -(E_h^{\pm} + \Gamma_h^{\pm}) f_h && (16)
\end{aligned}
$$

The proof of (16) is easily obtained by adding and subtracting the term

$$
-\sum_{i=1}^{n}(m_i h)\partial_h^{\pm i}(f_h(mh) - f_h(mh \mp h\mathbf{e}_i)),
$$

which is the same as $A_h^{\pm} f_h$, on the right hand side of (15). Moreover, for discrete monogenic polynomials of degree k, $M_k^{\pm} \in \mathcal{M}_k^{\pm}$, we have $\Gamma_h^{\pm} M_k^{\pm} = -kM_k^{\pm}$.

For all what follows, we introduce the difference operators

$$
\begin{aligned}
B_h^{\pm} &= \pm h \sum_{i=1}^{n} \partial_h^{\pm i}, && (17)\\
C_h^{\pm} f_h &= \sum_{i=1}^{n}(m_i h)\mathbf{e}_i f_h(\cdot \mp h\mathbf{e}_i) && (18)\\
R_{h,r}^{\pm} &= rI + E_h^{\pm} - A_h^{\pm}, && (19)\\
V_{h,r}^{\pm} &= R_{h,r}^{\pm} + \frac{1}{2} B_h^{\pm}. && (20)
\end{aligned}
$$

where I is the identity operator and r a real number.

From the identity

$$
\begin{aligned}
\left((mh)D_h^{\pm} + D_h^{\pm}(mh)\right) f_h &= -2\sum_{j=1}^{n}(m_j h)\partial_h^{\pm j} f_h - n f_h\\
&= -2(E_h^{\pm} - A_h^{\pm}) f_h - n f_h\\
&= -2R_{h,n/2}^{\pm} f_h
\end{aligned}
$$

we get

$$
\left(D_h^{\pm}(mh)\right) f_h = (-2R_{h,n/2}^{\pm} + E_h^{\pm} + \Gamma_h^{\pm}) f_h, \qquad (21)
$$

by applying identity (16).

Now, we will show some important facts regarding our difference operators.

Proposition 3.1 *For a lattice function $f_h : \mathbb{R}_h^n \to C\ell_{0,n}$, we have*

$$
D_h^{\pm} E_h^{\pm} f_h = D_h^{\pm} f_h + E_h^{\pm} D_h^{\pm} f_h.
$$

Proof:
Starting from the definition, we can split $D_h^\pm E_h^\pm f_h$ in the sum

$$(D_h^\pm E_h^\pm f_h)(mh) \quad = \quad I_1^\pm(mh) + I_2^\pm(mh) \tag{22}$$

with

$$I_1^\pm(mh) = \sum_{i=1}^n \mathbf{e}_i \partial_h^{\pm i} \left((m_i h)(\partial_h^{\pm i} f_h)(mh \mp h\mathbf{e}_i)\right)$$

and

$$I_2^\pm(mh) = \sum_{j=1}^n \sum_{i\neq j} \mathbf{e}_j \partial_h^{\pm j} \left((m_i h)(\partial_h^{\pm i} f_h)(mh \mp h\mathbf{e}_i)\right).$$

Applying the product rule for finite differences (2) in $I_1^\pm$, we obtain

$$\begin{aligned}
I_1^\pm(mh) &= \sum_{i=1}^n \mathbf{e}_i \left((\partial_h^{\pm i} f_h)(mh) + (m_i h)(\partial_h^{\pm 2\mathbf{e}_i} f_h)(mh \mp h\mathbf{e}_i)\right) \\
&= (D_h^\pm f_h)(mh) + \sum_{i=1}^n \mathbf{e}_i (m_i h)(\partial_h^{\pm 2\mathbf{e}_i} f_h)(mh \mp h\mathbf{e}_i).
\end{aligned}$$

On the other hand,

$$I_2^\pm(mh) = \sum_{j=1}^n \mathbf{e}_j \sum_{i\neq j} (m_i h) \Big((\partial_h^{\pm i} \partial_h^{\pm j} f_h)(mh \mp h\mathbf{e}_i)\Big).$$

Thus, we have

$$\begin{aligned}
(D_h^\pm E_h^\pm f_h)(mh) &= (D_h^\pm f_h)(mh) + \sum_{i,j=1}^n \mathbf{e}_j (m_i h)(\partial_h^{\pm i} \partial_h^{\pm j} f_h)(mh \mp h\mathbf{e}_i) \\
&= (D_h^\pm f_h)(mh) + (E_h^\pm D_h^\pm f_h)(mh).
\end{aligned}$$

■

Proposition 3.2 *For a lattice function $f_h : \mathbb{R}_h^n \to C\ell_{0,n}$, we have*

$$D_h^\pm((mh) f_h) = -2V_{h,n/2}^\pm f_h - (mh) D_h^\pm f_h \tag{23}$$

Proof: Using the product rule for finite differences (2) and the identity $-2m_i h = \mathbf{e}_i(mh) + (mh)\mathbf{e}_i, i = 1, \ldots, n$, we get

$$\begin{aligned}
D_h^\pm((mh) f_h(mh)) &= -\sum_{i=1}^n f_h(mh \pm h\mathbf{e}_i) - 2\sum_{i=1}^n (m_i h)\partial_h^{\pm i} f_h(mh) \\
&\quad -(mh)(D_h^\pm f_h)(mh) \\
&= -n f_h(mh) - \sum_{i=1}^n (f_h(mh \pm h\mathbf{e}_i) - f_h(mh)) -
\end{aligned}$$

$$
\begin{aligned}
& -2(E_h^{\pm} f_h - A_h^{\pm} f_h)(mh) - (mh)(D_h^{\pm} f_h)(mh) \\
= \;& -n f_h(mh) - 2(E_h^{\pm} f_h - A_h^{\pm} f_h)(mh) - \\
& -(B_h^{\pm} f_h)(mh) - (mh)(D_h^{\pm} f_h)(mh) \\
= \;& -2(R_{h,n/2}^{\pm} f_h + \frac{1}{2} B_h^{\pm} f_h)(mh) - (mh)(D_h^{\pm} f_h)(mh) \\
= \;& -2(V_{h,n/2}^{\pm} f_h)(mh) - (mh)(D_h^{\pm} f_h)(mh).
\end{aligned}
$$

■

From Proposition 3.2 and from the commutation properties $D_h^{\pm} A_h^{\pm} = A_h^{\pm} D_h^{\pm}$ and $D_h^{\pm} B_h^{\pm} = B_h^{\pm} D_h^{\pm}$ follow the operator relations

$$D_h^{\pm} R_{h,r}^{\pm} = R_{h,r+1}^{\pm} D_h^{\pm}, \tag{24}$$

$$D_h^{\pm} V_{h,r}^{\pm} = V_{h,r+1}^{\pm} D_h^{\pm}. \tag{25}$$

Combining Proposition 3.2 with operator relation (25), we have for $M_{k-s}^{\pm} \in \mathcal{M}_{k-s}^{\pm}$,

$$
\begin{aligned}
(D_h^{\pm})^2((mh)^2 M_{k-s}^{\pm}) &= D_h^{\pm} \left(-2V_{h,n/2}^{\pm}((mh) M_{k-s}^{\pm}) + 2(mh) V_{h,n/2}^{\pm} M_{k-s}^{\pm} \right) \\
&= (-2)^2 \left(V_{h,n/2+1}^{\pm} V_{h,n/2}^{\pm} M_{k-s}^{\pm} - V_{h,n/2}^{\pm} V_{h,n/2}^{\pm} M_{k-s}^{\pm} \right) \\
&= (-2)^2 V_{h,n/2}^{\pm} M_{k-s}^{\pm}. \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
(D_h^{\pm})^3((mh)^3 M_{k-s}^{\pm}) &= (D_h^{\pm})^2 \Big(-2V_{h,n/2}^{\pm}((mh)^2 M_{k-s}^{\pm}) \\
&\quad +2(mh) V_{h,n/2}^{\pm}((mh)^2 M_{k-s}^{\pm}) \Big) \\
&= (-2)^3 V_{h,n/2+2}^{\pm} V_{h,n/2}^{\pm} M_{k-s}^{\pm} \\
&\quad +2(D_h^{\pm})^2 \left((mh) V_{h,n/2}^{\pm}((mh)^2 M_{k-s}^{\pm}) \right) \\
&= (-2)^3 \Big(V_{h,n/2+2}^{\pm} V_{h,n/2}^{\pm} - V_{h,n/2+1}^{\pm} V_{h,n/2}^{\pm} \\
&\quad + V_{h,n/2}^{\pm} V_{h,n/2}^{\pm} \Big) M_{k-s}^{\pm} \\
&= (-2)^3 V_{h,n/2+1}^{\pm} V_{h,n/2}^{\pm} M_{k-s}^{\pm} \qquad (27)
\end{aligned}
$$

Continuing this procedure, we obtain by recursion

$$(D_h^{\pm})^s((mh)^s M_{k-s}^{\pm}) = (-2)^s V_{h,n/2+s-2}^{\pm} V_{h,n/2+s-3}^{\pm} \cdots V_{h,n/2}^{\pm} M_{k-s}^{\pm}. \tag{28}$$

From this follows also $(mh)^s M_{k-s}^{\pm} \in \ker(D_h^{\pm})^{s+1}$. Formula (28) gives us a motivation to find explicit formulae for the polynomials $M_{k-s}^{\pm}$. To this end we need an explicit formula for the inverse of the iterated composite operator $V_{h,n/2+s-2}^{\pm} V_{h,n/2+s-3}^{\pm} \cdots V_{h,n/2}^{\pm}$. This means that we have to find an explicit formula for the inverse of the operator $V_{h,r}^{\pm}$. Unfortunately, we are only able to get an explicit formula for the operator $R_{h,r}^{\pm}$ (c.f. [9]).

Theorem 3.5 *For a lattice function $f_h : \mathbb{R}_h^n \to C\ell_{0,n}$ and for $r > 0$, the difference operator $J_{h,r}^{\pm}$ defined by*

$$(J_{h,r}^{\pm} f_h)(mh) = \sum_{th \in [0,1]_h^{\pm}} h d_h^{\pm} \left((th \mp h)_{\mp}^{(r-1)} f_h((th)(mh)) \right)$$

satisfies

$$J_{h,r}^{\pm} R_{h,r}^{\pm} = I = R_{h,r}^{\pm} J_{h,r}^{\pm}.$$

Hereby we denote $[0,1]_h^+ = [0,1)_h, [0,1]_h^- = (0,1]_h$, and

$$(d_h^{\pm} g)(th) := \mp \frac{g(th) - g(th \pm h)}{h}.$$

Proof: (c.f. [9]) For $f_h : \mathbb{R}_h^n \to C\ell_{0,n}$ and $r > 0$,

$$f_h(mh) = \sum_{th \in [0,1]_h} h d_h^{\pm} \left((th \mp h)_{\mp}^{(r)} f_h((th)(mh)) \right)$$

By a direct calculation,

$$\begin{aligned} & d_h^{\pm} \left((th \mp h)_{\mp}^{(r)} f_h((th)(mh)) \right) \\ = \; & r(th \mp h)_{\mp}^{(r-1)} f_h((th)(mh)) + (th)_{\mp}^{(r)} (d_h^{\pm} f_h)((th)(mh)) \\ = \; & (th \mp h)_{\mp}^{(r-1)} \left(r f_h((th)(mh)) + th(d_h^{\pm} f_h)((th)(mh)) \right) \end{aligned}$$

On the other hand, applying the difference version of the chain rule and the relation $\sum_{i=1}^{n} (m_i h) \partial_h^{\pm i} = E_h^{\pm} - A_h^{\pm}$, we obtain

$$\begin{aligned} th(d_h^{\pm} f_h)((th)(mh)) &= \sum_{i=1}^{n} (th)(m_i h)(\partial_{th}^{\pm i} f_h)((th)(mh)) \\ &= (E_{th}^{\pm} f_h)((th)(mh)) - (A_{th}^{\pm} f_h)((th)(mh)) \end{aligned}$$

Therefore,

$$\begin{aligned} f_h(mh) &= r(J_{h,r}^{\pm} f_h)(mh) + (E_h^{\pm} J_{h,r}^{\pm} f_h)(mh) - (A_h^{\pm} J_{h,r}^{\pm} f_h)(mh) \\ &= (R_{h,r}^{\pm} J_{h,r}^{\pm} f_h)(mh). \end{aligned}$$

From the above two identities and the definitions of $R_{h,r}^{\pm}$ and $J_{h,r}^{\pm}$, we get

$$J_{h,r}^{\pm} R_{h,r}^{\pm} = I = R_{h,r}^{\pm} J_{h,r}^{\pm}.$$

■

Now, the construction of the inverse for $V_{h,r}^{\pm}$ seems to be obvious. But, the obvious choice

$$(W_{h,r}^{\pm} f_h)(mh) = \sum_{th \in [0,1]_h^{\pm}} h d_h^{\pm} \left((th)_{\mp}^{(r-1)} f_h((th)(mh)) \right)$$

is not an inverse of $V_{h,r}^{\pm}$, which can be easily checked in the following way.

If we use the same technique as above, we obtain

$$f_h(mh) = \sum_{th\in[0,1]_h^{\pm}} hd_h^{\pm}\left((th)_{\mp}^{(r-1)} f_h((th)(mh))\right)$$

and by direct calculation

$$\begin{aligned}
& d_h^{\pm}\left((th)_{\mp}^{(r)} f_h((th)(mh))\right) \\
= \;& r(th)_{\mp}^{(r-1)} f_h(thx) + (th \pm h)_{\mp}^{(r)} (d_h^{\pm} f_h)((th)(mh)) \\
= \;& (th)_{\mp}^{(r-1)} \left(r f_h((th)(mh)) + (th \pm h)(d_h^{\pm} f_h)((th)(mh))\right)
\end{aligned}$$

On the other hand, applying the difference version of the chain rule and the relation $\sum_{i=1}^{n}(m_i h)\partial_h^{\pm i} = E_h^{\pm} - A_h^{\pm}$, we obtain

$$\begin{aligned}
(th \pm h)(d_h^{\pm} f_h)((th)(mh)) &= \sum_{i=1}^{n}((th)(m_i h) \pm h m_i)(\partial_{th}^{\pm i} f_h)((th)(mh)) \\
&= (E_{th}^{\pm} f_h)((th)(mh)) - (A_{th}^{\pm} f_h)((th)(mh)) \\
&\quad \pm h \sum_{i=1}^{n} m_i h (\partial_{th}^{\pm i} f_h)((th)(mh)),
\end{aligned}$$

but

$$\begin{aligned}
\pm h \sum_{i=1}^{n}(m_i h)(\partial_{th}^{\pm i} f_h)((th)(mh)) &\neq \pm h \sum_{i=1}^{n}(\partial_{th}^{\pm i} f_h)((th)(mh)) \\
&= (B_{th}^{\pm} f_h)((th)(mh)).
\end{aligned}$$

3.4 Difference operator calculus

Now we will establish some properties for our difference operators introduced in Section 3.3.

Using the difference properties

$$(m_i h)\partial_h^{\pm i}(mh \mp h\mathbf{e}_i)_{\mp}^{(\alpha)} = \alpha_i (mh)_{\mp}^{(\alpha)}$$

and

$$(m_i h)\partial_h^{\pm i}\partial_h^{\mp i}(mh \mp h\mathbf{e}_i)_{\mp}^{(\alpha)} = (\alpha_i - 1)(m_i h)\partial_h^{\pm i}(mh \mp h\mathbf{e}_i)_{\mp}^{(\alpha)}$$

we obtain by direct calculation the following formulae for homogeneous polynomials of degree k, $P_k^{\pm} \in \Pi_k^{\pm}$,

$$B_h^{\pm} P_k^{\pm} = \pm k h P_k^{\pm}, \tag{29}$$

$$A_h^{\pm} P_k^{\pm} = \frac{kh^2}{1 \pm h} P_k^{\pm}, \tag{30}$$

$$R_{h,r}^{\pm} P_k^{\pm} = \left(r + k - \frac{kh^2}{1 \pm h}\right) P_k^{\pm}, \tag{31}$$

$$V_{h,r}^{\pm} P_k^{\pm} = \left(r + \left(1 \pm \frac{h}{2}\right) k - \frac{kh^2}{1 \pm h}\right) P_k^{\pm}. \tag{32}$$

Now, using the difference rules (2) and (3), we get

$$B_h^{\pm}((mh)f_h) = (mh)B_h^{\pm}f_h + h\mathbf{1}^{\pm}f_h(mh) + h^2D_h^{\pm}f_h, \tag{33}$$

$$C_h^{\pm}((mh)f_h) = (mh)C_h^{\pm}f_h - E_h^{\pm}f_h, \tag{34}$$

$$A_h^{\pm}((mh)f_h) = (mh)A_h^{\pm}f_h \mp hC_h^{\pm}f_h, \tag{35}$$

$$E_h^{\pm}((mh)f_h) = (mh)E_h^{\pm}f_h + C_h^{\pm}f_h, \tag{36}$$

$$R_{h,r}^{\pm}((mh)f_h) = (mh)R_{h,r}^{\pm}f_h + (1\pm h)C_h^{\pm}f_h, \tag{37}$$

$$\begin{aligned} V_{h,r}^{\pm}((mh)f_h) &= (mh)V_{h,r}^{\pm}f_h + (1\pm h)C_h^{\pm}f_h + \\ &+ \frac{1}{2}(h\mathbf{1}^{\pm}f_h(mh) + h^2D_h^{\pm}f_h), \end{aligned} \tag{38}$$

where $\mathbf{1}^{\pm} := \pm\sum_{i=1}^{n}\mathbf{e}_i$.

As a direct consequence of formulae (32) and (36), we obtain for the discrete homogeneous monogenic polynomials of degree k, $M_k^{\pm}\in\mathcal{M}_k^{\pm}$,

$$\begin{aligned} \Gamma_h^{\pm}((mh)M_k^{\pm}) &= -E_h^{\pm}((mh)M_k^{\pm}) - (mh)D_h^{\pm}((mh)M_k^{\pm}) \\ &= \left(n + k - \frac{2kh^2}{1\pm h} \pm hk\right)(mh)M_k^{\pm} - C_h^{\pm}M_k^{\pm} \end{aligned} \tag{39}$$

by applying Theorem 3.2 and relation (16).

Moreover, using identities (21), (29),(31),(32),(16) and Proposition 3.2, we obtain the relation

$$\begin{aligned} \left(D_h^{\pm}(mh)\right)((mh)M_k^{\pm}) &= (-2R_{h,n/2}^{\pm} + E_h^{\pm} + \Gamma_h^{\pm})((mh)M_k^{\pm}) \\ &= \pm hk(mh)M_k^{\pm} - (2\pm 2h)C_h^{\pm}M_k^{\pm}. \end{aligned} \tag{40}$$

Applying relations (36) and (34) we have

$$E_h^{\pm}((mh)^2M_k^{\pm}) = k(mh)^2M_k^{\pm} + 2(mh)C_h^{\pm}M_k^{\pm} - kM_k^{\pm}. \tag{41}$$

From Theorem 3.2 and formulae (33) and (40), we get

$$D_h^{\pm}((mh)^2M_k^{\pm}) = -(2\pm 2h)C_h^{\pm}M_k^{\pm} + h\mathbf{1}^{\pm}M_k^{\pm}. \tag{42}$$

Using (16) and formulae (41) and (42), we obtain

$$\Gamma_h^{\pm}((mh)^2M_k^{\pm}) = -k(mh)^2M_k^{\pm} \pm 2(mh)C_h^{\pm}M_k^{\pm} + kM_k^{\pm} - h\mathbf{1}^{\pm}(mh)M_k^{\pm}. \tag{43}$$

Applying recursively our formulae (33)-(38), it is possible to obtain explicit formulae for $D_h^{\pm}((mh)^sM_k^{\pm})$, $E_h^{\pm}((mh)^sM_k^{\pm})$ and $\Gamma_h^{\pm}((mh)^sM_k^{\pm})$ by induction.

3.5 Homogeneous powers

Contrary to the continuous case, the classical product between the variable mh and the homogeneous polynomial $P_k^{\pm}$ is not homogeneous. However, applying formulae (6) and (7) proved in Theorem 3.2, we can say that the product $(mh)P_k^{\pm}$ can be expressed as a linear combination of homogeneous polynomials up to degree $k+1$. On the other hand, the powers $x^s = (mh)^s$ are not homogeneous. For this purpose, we will introduce the discrete analogues of x^s in the following way:

Starting from the multi-index factorial powers of degree $|\alpha|$, $(mh)_{\mp}^{(\alpha)}$, we introduce the polynomials $H_s^{\pm}, s \in \mathbb{N}$ by

$$H_s^{\pm}(mh) = \begin{cases} \sum_{|\alpha|=k} \frac{(-1)^k k!}{\alpha!} (mh)_{\mp}^{(2\alpha)} & if\ s = 2k, k \in \mathbb{N}_0 \\ \sum_{|\alpha|=k} \sum_{i=1}^{n} \frac{(-1)^k k!}{\alpha!} (mh)_{\mp}^{(2\alpha+\mathbf{e}_i)} \mathbf{e}_i & if\ s = 2k+1, k \in \mathbb{N}_0. \end{cases} \tag{44}$$

As a direct consequence of the identity

$$x^s = \begin{cases} (-1)^k |x|^{2k} & if\ s = 2k, k \in \mathbb{N}_0 \\ x x^{2k} & if\ s = 2k+1, k \in \mathbb{N}_0 \end{cases}$$

we can conclude by Lemma 3.3 and by the multinomial theorem, that the polynomials $H_s^{\pm}$ give rise to homogeneous polynomials of degree s which approximate the powers $x^s = (mh)^s$ for small mesh width $h > 0$.

As a direct consequence, the operator formulae proved in Subsection 3.4 are fulfilled for the powers $H_s^{\pm}$ and, moreover, by direct calculation, we obtain the additional properties

$$C_h^{\pm} H_s^{\pm} = H_{s+1}^{\pm},$$

and

$$D_h^{\pm} H_s^{\pm} = -s H_{s-1}^{\pm}.$$

Let us remark that the term $C_h^{\pm} H_s^{\pm}$ is the discrete version of the multiplication $x x^s$ in the continuous case.

4 A discrete harmonic Fischer decomposition

According to the classical theory of the finite differences, the usual approximation of the Laplacian is given by

$$\begin{aligned} (\Delta_h u)(mh) &= \sum_{i=1}^{n} \frac{u(mh + h\mathbf{e}_i) + u(mh - h\mathbf{e}_i) - 2u(mh)}{h^2} \\ &= \sum_{i=1}^{n} (\partial_h^{\mp i} \partial_h^{\pm i} u)(mh). \end{aligned} \tag{45}$$

The first problem that arises now is that not all of our partial difference operators do commute in the certain sense (c.f. [5, 6]) and, moreover, we have

no factorization of the discrete Laplacian Δ_h by means of our difference Dirac operators considered above, that is $D_h^{\mp} D_h^{\pm} \neq -\mathbf{e}_0 \Delta_h$.

Let us restrict ourselves in this section to the case of quaternion-valued functions defined on lattices in $\mathbb{R}^3$.
Let us remark that the quaternionic variable mh is identified with the 4×4 matrix

$$mh = \begin{pmatrix} 0 & -m_1h & -m_2h & -m_3h \\ m_1h & 0 & -m_3h & m_2h \\ m_2h & m_3h & 0 & m_1h \\ m_3h & -m_2h & m_1h & 0 \end{pmatrix}.$$

In [7] for a lattice function $f_h : \mathbb{R}_h^3 \to \mathbb{H}$ given by

$$f_h = \sum_{i=0}^{3} f_h^i \mathbf{e}_i = f_h^0 \mathbf{e}_0 + \mathrm{Vec}\, f_h$$

a finite difference approximation of our Dirac operator was defined in the form

$$\begin{aligned} D_h^{-+} f_h &= \begin{pmatrix} 0 & -\partial_h^{-1} & -\partial_h^{-2} & -\partial_h^{-3} \\ \partial_h^{-1} & 0 & -\partial_h^{3} & \partial_h^{2} \\ \partial_h^{-2} & \partial_h^{3} & 0 & -\partial_h^{1} \\ \partial_h^{-3} & -\partial_h^{2} & \partial_h^{1} & 0 \end{pmatrix} \begin{pmatrix} f_h^0 \\ f_h^1 \\ f_h^2 \\ f_h^3 \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{div}_h^{-} \mathrm{Vec}\, f_h \\ \mathrm{grad}_h^{-} f_h^0 + \mathrm{curl}_h^{+} \mathrm{Vec}\, f_h \end{pmatrix} \end{aligned} \tag{46}$$

$$\begin{aligned} D_h^{+-} f_h &= \begin{pmatrix} 0 & -\partial_h^{1} & -\partial_h^{2} & -\partial_h^{3} \\ \partial_h^{1} & 0 & -\partial_h^{-3} & \partial_h^{-2} \\ \partial_h^{2} & \partial_h^{-3} & 0 & -\partial_h^{-1} \\ \partial_h^{3} & -\partial_h^{-2} & \partial_h^{-1} & 0 \end{pmatrix} \begin{pmatrix} f_h^0 \\ f_h^1 \\ f_h^2 \\ f_h^3 \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{div}_h^{+} \mathrm{Vec}\, f_h \\ \mathrm{grad}_h^{+} f_h^0 + \mathrm{curl}_h^{-} \mathrm{Vec}\, f_h \end{pmatrix} \end{aligned} \tag{47}$$

with $\mathrm{div}_h^{\pm} \mathrm{Vec}\, f_h = \sum_{i=1}^3 \partial_h^{\pm i} f_h^i$, $\mathrm{grad}_h^{\pm} f_h^0 = \sum_{i=1}^3 (\partial_h^{\pm i} f_h^0) \mathbf{e}_i$ and

$$\mathrm{curl}_h^{\pm} \mathrm{Vec}\, f_h = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_h^{\pm 1} & \partial_h^{\pm 2} & \partial_h^{\pm 3} \\ f_h^1 & f_h^2 & f_h^3 \end{vmatrix}.$$

In the latter form one can easily see the similarity with the usual Dirac operator

$$Df = \begin{pmatrix} -\mathrm{div}\mathrm{Vec}\, f \\ \mathrm{grad}\mathrm{Sc}\, f + \mathrm{curl}\mathrm{Vec}\, f \end{pmatrix}.$$

Using the discrete identities

$$\begin{aligned} \mathrm{div}_h^{\pm} \mathrm{curl}_h^{\pm} \mathrm{Vec}\, f_h &= 0 \\ \mathrm{curl}_h^{\pm} \mathrm{grad}_h^{\pm} f_h^0 &= \mathbf{0} \\ \mathrm{curl}_h^{\pm} \mathrm{curl}_h^{\mp} \mathrm{Vec}\, f_h &= -\Delta_h \mathrm{Vec}\, f_h + \mathrm{grad}_h^{\mp} \mathrm{div}_h^{\pm} \mathrm{Vec}\, f_h \end{aligned}$$

we obtain the following factorization of the discrete Laplacian

$$D_h^{+-} D_h^{-+} f_h = \begin{pmatrix} -\Delta_h f_h^0 \\ -\Delta_h \mathrm{Vec}\, f_h \end{pmatrix} = D_h^{-+} D_h^{+-} f_h. \tag{48}$$

Now, we are able to obtain a Fischer decomposition for the discrete Dirac operators D_h^{-+} and D_h^{+-}.

Using the fact that

$$\begin{aligned} D_h^{-+} (mh)_+^{(\alpha)} &= \begin{pmatrix} 0 \\ \mathrm{grad}_h^- (mh)_+^{(\alpha)} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \alpha_1 (mh)_+^{(\alpha - \mathbf{e}_1)} \\ \alpha_2 (mh)_+^{(\alpha - \mathbf{e}_2)} \\ \alpha_3 (mh)_+^{(\alpha - \mathbf{e}_3)} \end{pmatrix} \end{aligned}$$

as well as

$$\begin{aligned} D_h^{+-} (mh)_-^{(\alpha)} &= \begin{pmatrix} 0 \\ \mathrm{grad}_h^+ (mh)_-^{(\alpha)} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \alpha_1 (mh)_-^{(\alpha - \mathbf{e}_1)} \\ \alpha_2 (mh)_-^{(\alpha - \mathbf{e}_2)} \\ \alpha_3 (mh)_-^{(\alpha - \mathbf{e}_3)} \end{pmatrix} \end{aligned}$$

we can prove as in Lemma 3.4, the inclusion properties $D_h^{+-} \Pi_k^+ \subset \Pi_{k-1}^+$, $D_h^{-+} \Pi_k^- \subset \Pi_{k-1}^-$ and, moreover, replacing D_h^{+-} by D_h^+ and D_h^{-+} by D_h^- in the inner product (10), we obtain the Fischer decompositions:

Theorem 4.1 Fischer decomposition for D_h^{-+} and D_h^{+-}

Let $P_k^- \in \Pi_k^-$ (respectively, $P_k^+ \in \Pi_k^+$) then

$$P_k^- = \sum_{s=0}^{k-1} (mh)^s M_{k-s}^{-+}, \tag{49}$$

$$P_k^+ = \sum_{s=0}^{k-1} (mh)^s M_{k-s}^{+-}. \tag{50}$$

where each M_j^{-+}(respectively, M_j^{+-}) denotes a homogeneous discrete monogenic polynomial of degree j, that is, $M_j^{-+} \in \Pi_j^- \cap \ker D_h^{-+}$(respectively, $M_j^{+-} \in \Pi_j^+ \cap \ker D_h^{+-}$).

From the factorization property (48), we have

$$[(mh)^2 P_k^{\pm}, Q_k^{\pm}]_h = -[P_k^{\pm}, \Delta_h Q_k^{\pm}]_h,$$

which allows us to obtain the Fischer decomposition for the discrete Laplacian:

Theorem 4.2 Fischer decomposition for Δ_h

Let $P_k^{\pm} \in \Pi_k^{\pm}$ then

$$P_k^{\pm} = \sum_{2s \leq k} |mh|^{2s} \mathcal{H}_{k-2s}^{\pm},$$

where each $\mathcal{H}_j^{\pm}$ denotes a homogeneous discrete harmonic polynomial of degree j, that is, $\mathcal{H}_j^{\pm} \in \Pi_j^{\pm} \cap \ker \Delta_h$.

As a consequence of Theorem 4.1, we obtain Fischer decompositions which relate the discrete harmonic and the discrete monogenic polynomials.

Corollary 4.1 Fischer decomposition *Let $\mathcal{H}_k^{\pm} \in \Pi_k^{\pm} \cap \ker \Delta_h$ then*

$$\mathcal{H}_k^{-} = M_k^{-+} + (mh) M_{k-1}^{-+}, \tag{51}$$
$$\mathcal{H}_k^{+} = M_k^{+-} + (mh) M_{k-1}^{+-}. \tag{52}$$

where each M_j^{-+}(respectively, M_j^{+-}) denotes a homogeneous discrete monogenic polynomial of degree j, that is, $M_j^{-+} \in \Pi_j^{-} \cap \ker D_h^{-+}$(respectively, $M_j^{+-} \in \Pi_j^{+} \cap \ker D_h^{+-}$).

To define the Euler and Gamma operator E_h^{+-}, Γ_h^{+-} and E_h^{-+}, Γ_h^{-+} for the modified Dirac operators D_h^{+-} and D_h^{-+}, respectively, we start to calculate the products $(mh)D_h^{+-} f_h$ and $(mh)D_h^{+-} f_h$. By straightforward calculations we obtain

$$\begin{aligned}
(mh)D_h^{-+} f_h &= -\begin{pmatrix} \partial_h^{-1} f_h^0 & \partial_h^{-2} f_h^0 & \partial_h^{-3} f_h^0 \\ \partial_h^{-1} f_h^1 & \partial_h^{2} f_h^1 & \partial_h^{3} f_h^1 \\ \partial_h^{1} f_h^2 & \partial_h^{-2} f_h^2 & \partial_h^{3} f_h^2 \\ \partial_h^{1} f_h^3 & \partial_h^{2} f_h^3 & \partial_h^{-3} f_h^3 \end{pmatrix} \begin{pmatrix} m_1 h \\ m_2 h \\ m_3 h \end{pmatrix} \\
&+ \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1 h & m_2 h & m_3 h \\ \partial_h^{-1} & \partial_h^{-2} & \partial_h^{-3} \end{vmatrix} f_h^0 + \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1 h & m_3 h & m_2 h \\ \partial_h^{-1} & \partial_h^{3} & \partial_h^{2} \end{vmatrix} f_h^1 \\
&+ \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_3 \\ -m_2 h & -m_3 h & m_1 h \\ -\partial_h^{-2} & -\partial_h^{3} & \partial_h^{1} \end{vmatrix} f_h^2 \\
&+ \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_3 \\ m_3 h & m_2 h & m_1 h \\ \partial_h^{-3} & \partial_h^{2} & \partial_h^{1} \end{vmatrix} f_h^3.
\end{aligned} \tag{53}$$

and

$$(mh)D_h^{+-} f_h = -\begin{pmatrix} \partial_h^{1} f_h^0 & \partial_h^{2} f_h^0 & \partial_h^{3} f_h^0 \\ \partial_h^{1} f_h^1 & \partial_h^{-2} f_h^1 & \partial_h^{-3} f_h^1 \\ \partial_h^{-1} f_h^2 & \partial_h^{2} f_h^2 & \partial_h^{-3} f_h^2 \\ \partial_h^{-1} f_h^3 & \partial_h^{-2} f_h^3 & \partial_h^{3} f_h^3 \end{pmatrix} \begin{pmatrix} m_1 h \\ m_2 h \\ m_3 h \end{pmatrix}$$

$$+\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1h & m_2h & m_3h \\ \partial_h^1 & \partial_h^2 & \partial_h^3 \end{vmatrix} f_h^0 + \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1h & m_3h & m_2h \\ \partial_h^1 & \partial_h^{-3} & \partial_h^{-2} \end{vmatrix} f_h^1$$

$$+\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_3 \\ -m_2h & -m_3h & m_1h \\ -\partial_h^2 & -\partial_h^{-3} & \partial_h^{-1} \end{vmatrix} f_h^2$$

$$+\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ m_3h & m_2h & m_1h \\ \partial_h^3 & \partial_h^{-2} & \partial_h^{-1} \end{vmatrix} f_h^3. \tag{54}$$

Hence, we can define the difference Euler operators E_h^{-+} and E_h^{+-} as

$$(E_h^{-+}f_h)(mh) =$$
$$\begin{pmatrix} (\partial_h^{-1}f_h^0)(mh+h\mathbf{e}_1) & (\partial_h^{-2}f_h^0)(mh+h\mathbf{e}_2) & (\partial_h^{-3}f_h^0)(mh+h\mathbf{e}_3) \\ (\partial_h^{-1}f_h^1)(mh+h\mathbf{e}_1) & (\partial_h^{2}f_h^1)(mh-h\mathbf{e}_2) & (\partial_h^{3}f_h^1)(mh-h\mathbf{e}_3) \\ (\partial_h^{1}f_h^2)(mh-h\mathbf{e}_1) & (\partial_h^{-2}f_h^2)(mh+h\mathbf{e}_2) & (\partial_h^{3}f_h^2)(mh-h\mathbf{e}_3) \\ (\partial_h^{1}f_h^3)(mh-h\mathbf{e}_1) & (\partial_h^{2}f_h^3)(mh-h\mathbf{e}_2) & (\partial_h^{-3}f_h^3)(mh+h\mathbf{e}_3) \end{pmatrix} \begin{pmatrix} m_1h \\ m_2h \\ m_3h \end{pmatrix}$$

and

$$(E_h^{+-}f_h)(mh) =$$
$$\begin{pmatrix} (\partial_h^{1}f_h^0)(mh-h\mathbf{e}_1) & (\partial_h^{2}f_h^0)(mh-h\mathbf{e}_2) & (\partial_h^{3}f_h^0)(mh-h\mathbf{e}_3) \\ (\partial_h^{1}f_h^1)(mh-h\mathbf{e}_1) & (\partial_h^{-2}f_h^1)(mh+h\mathbf{e}_2) & (\partial_h^{-3}f_h^1)(mh+h\mathbf{e}_3) \\ (\partial_h^{-1}f_h^2)(mh+h\mathbf{e}_1) & (\partial_h^{2}f_h^2)(mh-h\mathbf{e}_2) & (\partial_h^{-3}f_h^2)(mh+h\mathbf{e}_3) \\ (\partial_h^{-1}f_h^3)(mh+h\mathbf{e}_1) & (\partial_h^{-2}f_h^3)(mh+h\mathbf{e}_2) & (\partial_h^{3}f_h^3)(mh-h\mathbf{e}_3) \end{pmatrix} \begin{pmatrix} m_1h \\ m_2h \\ m_3h \end{pmatrix}$$

The difference Gamma operators Γ_h^{-+} and Γ_h^{+-} are defined by

$$(\Gamma_h^{-+}f_h)(mh) =$$
$$= h\begin{pmatrix} (\partial_h^{-1}\partial_h^1 f_h^0)(mh) & (\partial_h^{-2}\partial_h^2 f_h^0)(mh) & (\partial_h^{-3}\partial_h^3 f_h^0)(mh) \\ (\partial_h^{-1}\partial_h^1 f_h^1)(mh) & -(\partial_h^{-2}\partial_h^2 f_h^1)(mh) & -(\partial_h^{-3}\partial_h^3 f_h^1)(mh) \\ -(\partial_h^{-1}\partial_h^1 f_h^2)(mh) & (\partial_h^{-2}\partial_h^2 f_h^2)(mh) & -(\partial_h^{-3}\partial_h^3 f_h^2)(mh) \\ -(\partial_h^{-1}\partial_h^1 f_h^3)(mh) & -(\partial_h^{-2}\partial_h^2 f_h^3)(mh) & (\partial_h^{-3}\partial_h^3 f_h^3)(mh) \end{pmatrix} \begin{pmatrix} m_1h \\ m_2h \\ m_3h \end{pmatrix}$$
$$-\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1h & m_2h & m_3h \\ \partial_h^{-1} & \partial_h^{-2} & \partial_h^{-3} \end{vmatrix} f_h^0(mh) - \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1h & m_3h & m_2h \\ \partial_h^{-1} & \partial_h^{3} & \partial_h^{2} \end{vmatrix} f_h^1(mh)$$
$$-\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_3 \\ -m_2h & -m_3h & m_1h \\ -\partial_h^{-2} & -\partial_h^{3} & \partial_h^{1} \end{vmatrix} f_h^2(mh)$$
$$-\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ m_3h & m_2h & m_1h \\ \partial_h^{-3} & \partial_h^{2} & \partial_h^{1} \end{vmatrix} f_h^3(mh)$$

and

$$(\Gamma_h^{+-}f_h)(mh) =$$

$$
\begin{aligned}
&= -h \begin{pmatrix} (\partial_h^{-1}\partial_h^1 f_h^0)(mh) & (\partial_h^{-2}\partial_h^2 f_h^0)(mh) & (\partial_h^{-3}\partial_h^3 f_h^0)(mh) \\ (\partial_h^{-1}\partial_h^1 f_h^1)(mh) & -(\partial_h^{-2}\partial_h^2 f_h^1)(mh) & -(\partial_h^{-3}\partial_h^3 f_h^1)(mh) \\ -(\partial_h^{-1}\partial_h^1 f_h^2)(mh) & (\partial_h^{-2}\partial_h^2 f_h^2)(mh) & -(\partial_h^{-3}\partial_h^3 f_h^2)(mh) \\ -(\partial_h^{-1}\partial_h^1 f_h^3)(mh) & -(\partial_h^{-2}\partial_h^2 f_h^3)(mh) & (\partial_h^{-3}\partial_h^3 f_h^3)(mh) \end{pmatrix} \begin{pmatrix} m_1 h \\ m_2 h \\ m_3 h \end{pmatrix} \\
&- \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1 h & m_2 h & m_3 h \\ \partial_h^1 & \partial_h^2 & \partial_h^3 \end{vmatrix} f_h^0(mh) - \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_2 & \mathbf{e}_3 \\ m_1 h & m_3 h & m_2 h \\ \partial_h^1 & \partial_h^{-3} & \partial_h^{-2} \end{vmatrix} f_h^1(mh) \\
&- \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_3 \\ -m_2 h & -m_3 h & m_1 h \\ -\partial_h^2 & -\partial_h^{-3} & \partial_h^{-1} \end{vmatrix} f_h^2(mh) \\
&- \begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ m_3 h & m_2 h & m_1 h \\ \partial_h^3 & \partial_h^{-2} & \partial_h^{-1} \end{vmatrix} f_h^3(mh).
\end{aligned}
$$

As in Subsection 3.3, we have $(mh)D_h^{-+} = -E_h^{-+} - \Gamma_h^{-+}$ (respectively, $(mh)D_h^{+-} = -E_h^{+-} - \Gamma_h^{+-}$) and the polynomials $P_k^{\pm} \in \Pi_k^{\pm}$ satisfy $E_h^{-+}P_k^- = kP_k^-$, (respectively, $E_h^{+-}P_k^+ = kP_k^+$). Moreover, if $P_k^- \in \ker D_h^{-+}$ (respectively, $P_k^+ \in \ker D_h^{+-}$) then we have $\Gamma_h^{-+}P_k^- = -kP_k^-$, (respectively, $\Gamma_h^{+-}P_k^+ = -kP_k^+$).

Like in Theorem 3.2 we can prove the operator property $D_h^{-+}E_h^{-+} = I + E_h^{-+}D_h^{-+}$ (respectively, $D_h^{+-}E_h^{+-} = I + E_h^{+-}D_h^{+-}$). In the same way we get analogous relations to the ones presented in Subsection 3.3 and in Subsection 3.4. In addition it is also possible define the discrete versions of the quaternionic powers $(mh)^s$ with respect to our difference Dirac operators D_h^{-+} and D_h^{+-}, using a similar construction as in Subsection 3.5.

At this point it would be interesting to know if the operator setting we discussed here for the quaternionic case has an equivalent operator setting in the general case of Clifford algebras. Up to know it is not known, but we will discuss it in a forthcoming paper [4].

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